# Arab Open University
# Faculty of Computer Studies
# MT101 - General Mathematics

# Chapter 2

Functions and Graphs

# SECTION 2.1 The Rectangular Coordinate System and Graphing Utilities

The **distance** between two points $(x_1, y_1)$ and $(x_2, y_2)$ in a rectangular coordinate system is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The **midpoint** between the points is given by:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

**Example 1:**

Given $(-4, -5)$ and $(6, -1)$, find the distance between the points and the midpoint of the line segment between the points.

$$d = \sqrt{[6 - (-4)]^2 + [-1 - (-5)]^2}$$

$$d = \sqrt{(10)^2 + (4)^2} = \sqrt{116} = 2\sqrt{29}$$

$$M = \left(\frac{-4 + 6}{2}, \frac{-5 + (-1)}{2}\right) = (1, -3)$$

## Graphing an equation:

To graph an equation in a rectangular coordinate system, plot several solutions to the equation to form a general outline of the curve. Then connect the points to form a smooth line or curve.

## Determining *x*- and *y*-intercepts:

To find an $x$-intercept $(a, 0)$ of the graph of an equation, substitute 0 for $y$ and solve for $x$.

To find a $y$-intercept $(0, b)$ of the graph of an equation, substitute 0 for $x$ and solve for $y$.

**Example 2:**

Graph $x = y^2 - 4$.

| $x$ | 5 | 0 | −3 | −4 | −3 | 0 | 5 |
|---|---|---|---|---|---|---|---|
| $y$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |

***x*-intercept:**

$x = y^2 - 4$

$x = (0)^2 - 4$

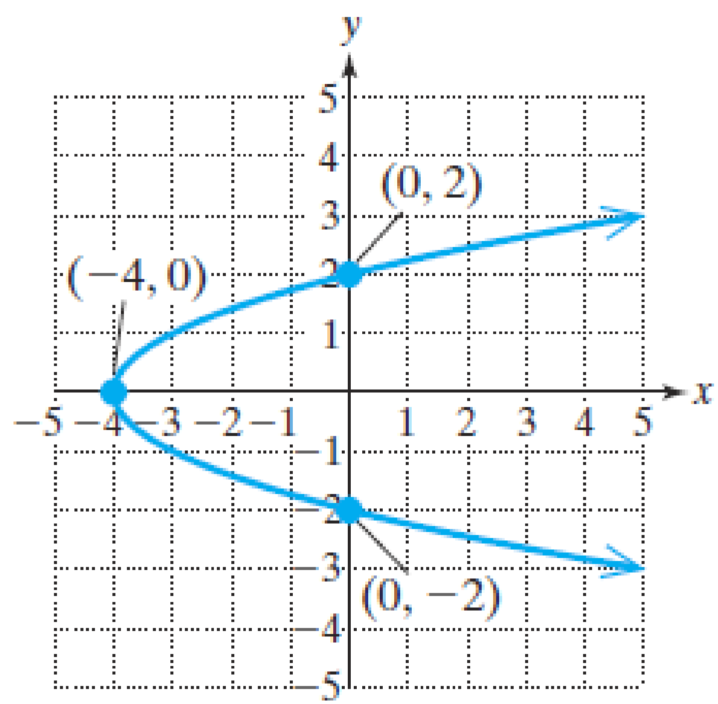
$x = -4$

***y*-intercepts:**

$x = y^2 - 4$

$0 = y^2 - 4$

$y = 2$ or $y = -2$

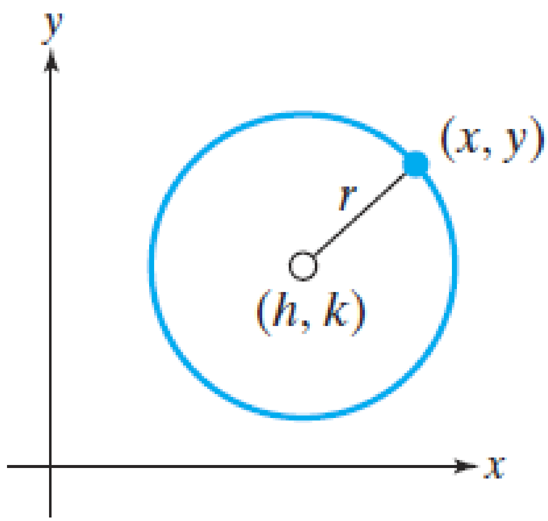


# SECTION 2.2 Circles

A **circle** is the set of all points in a plane that are equidistant from a fixed point called the **center.** The fixed distance between the center and any point on the circle is called the **radius** of the circle.

The **standard form** of an equation of a circle with radius $r$ and center $(h, k)$ is

$$(x - h)^2 + (y - k)^2 = r^2$$



**Example 1:**

Write the standard form of an equation of the circle whose center is $\left(3, -\frac{1}{2}\right)$ and whose radius is 5.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 3)^2 + \left[y - \left(-\frac{1}{2}\right)\right]^2 = (5)^2$$

$$(x - 3)^2 + \left(y + \frac{1}{2}\right)^2 = 25$$

An equation of a circle written in the form $x^2 + y^2 + Ax + By + C = 0$ is called the **general form** of an equation of a circle.

**Writing an equation in standard form:**

By completing the square, we can write an equation of a circle given in general form as an equation in standard form. The purpose is to identify the center and radius from the standard form.

**Example 2:**

Write the equation of the circle in standard form. Then identify the center and radius.

$$x^2 + y^2 - 12x + 8y + 5 = 0$$

$$x^2 - 12x + 36 + y^2 + 8y + 16 = -5 + 36 + 16$$

$$(x - 6)^2 + (y + 4)^2 = 47$$

The center is $(6, -4)$ and the radius is $\sqrt{47}$.

# SECTION 2.3 Functions and Relations

A set of ordered pairs $(x, y)$ is called a **relation** in $x$ and $y$. The set of $x$ values is the **domain** of the relation, and the set of $y$ values is the **range** of the relation.

Given a relation in $x$ and $y$, we say that $y$ **is a function of** $x$ if for each value of $x$ in the domain, there is exactly one value of $y$ in the range.

**Example 1:**

Given the relation {(8, 2), (3, 2), (9, 5)},

The domain is {8, 3, 9}.

The range is {2, 5}.

**Example 2:**

same *x*

The relation {(4, 6), (3, 5), (4, 1) is *not* a function.

different *y*

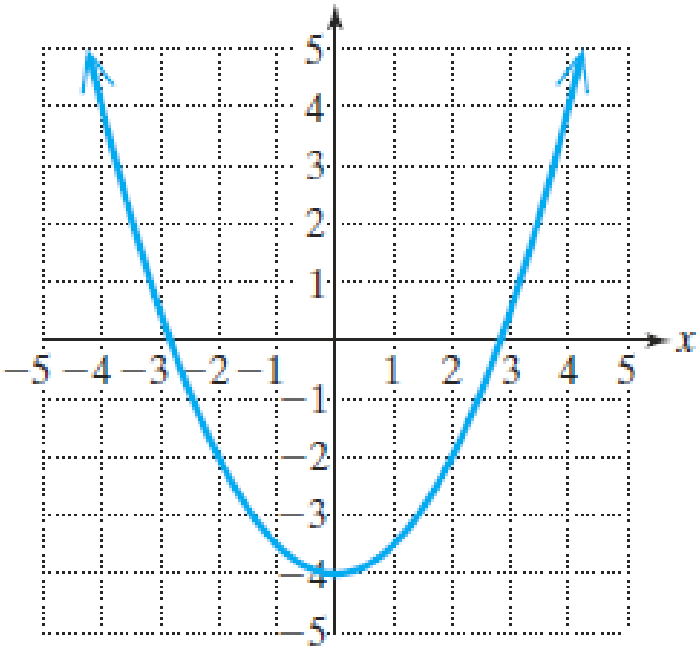
The relation {(3, 5), (2, 1), (9, 0)} is a function because no two ordered pairs have the same *x* value but different *y* values.

**Vertical line test:**

The graph of a relation defines $y$ as a function of $x$ if no vertical line intersects the graph in more than one point.

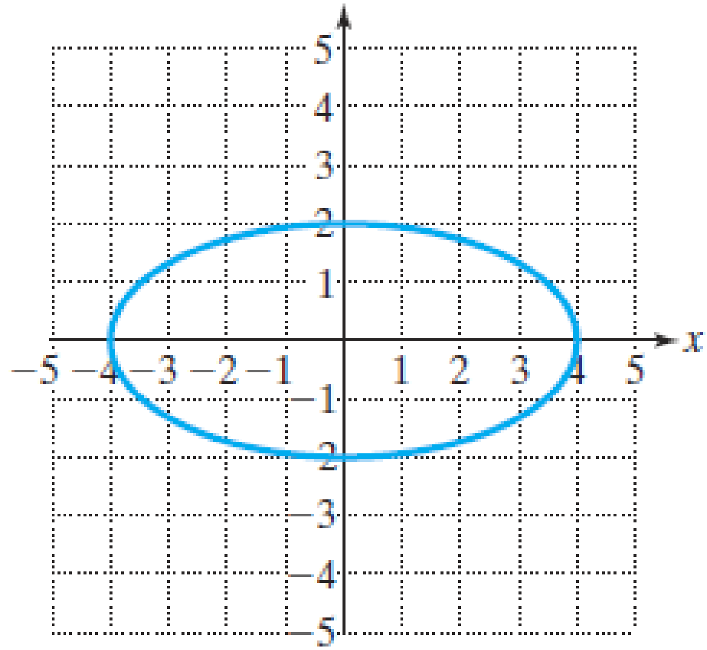
**Example 3:**

$y$ Function




Domain: $(-\infty, \infty)$
Range: $[-4, \infty)$

$y$ Not a function




Domain: $[-4, 4]$
Range: $[-2, 2]$

## Evaluating a function for different values of *x*:

A function may be evaluated at different values of $x$ by using substitution.

## Example 4:

Given $f(x) = 2x^2 + 3x$,

$$f(2) = 2(2)^2 + 3(2) = 14$$

$$\begin{aligned} f(x + 4) &= 2(x + 4)^2 + 3(x + 4) \\ &= 2(x^2 + 8x + 16) + 3x + 12 \\ &= 2x^2 + 19x + 44 \end{aligned}$$

**Determining *x*- and *y*-intercepts:**

Given a function defined by $y = f(x)$,

- The $x$-intercept(s) are the real solutions to $f(x) = 0$.
- The $y$-intercept is given by $f(0)$.

**Example 5:**

Given $f(x) = |x| - 2$,

- To find the $x$-intercept(s), substitute 0 for $f(x)$:

$$0 = |x| - 2$$

$$x = 2 \quad \text{or} \quad x = -2$$

  The $x$-intercepts are $(2, 0)$, $(-2, 0)$.

- To find the $y$-intercept, evaluate $f(0)$.

$$f(0) = |0| - 2 = -2$$

  The $y$-intercept is $(0, -2)$.

## Determining domain from $y = f(x)$:

Given $y = f(x)$, the domain of $f$ is the set of real numbers $x$ that when substituted into the function produce a real number. This excludes

- Values of $x$ that make the denominator zero.
- Values of $x$ that make a radicand negative within an even-indexed root.

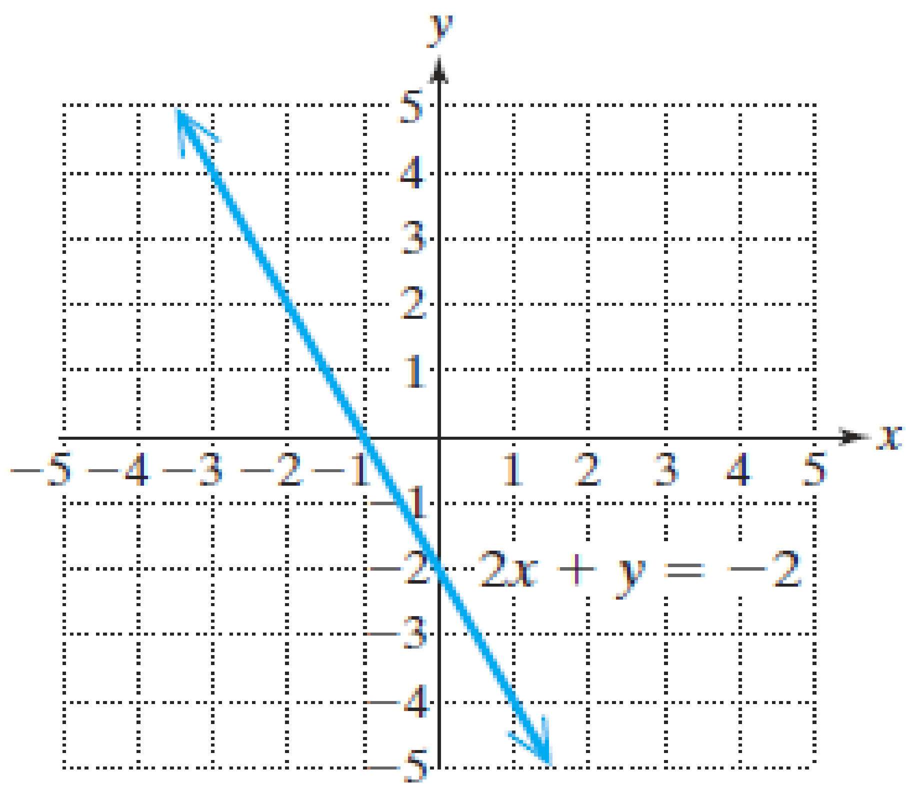
## Example 6:

- Given $f(x) = \dfrac{x+5}{x-3}$, the domain is $(-\infty, 3) \cup (3, \infty)$.
- Given $g(x) = \sqrt{x-3}$, the domain is $[3, \infty)$.

# SECTION 2.4 Linear Equations in Two Variables and Linear Functions

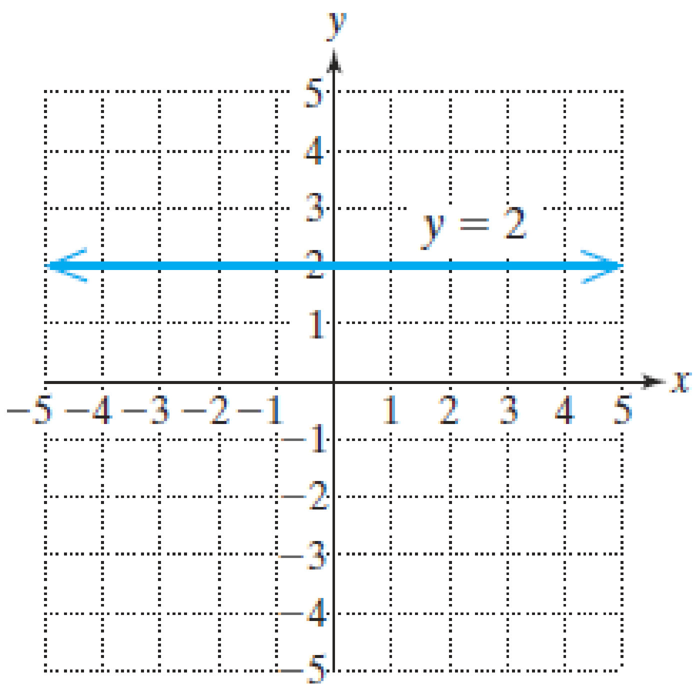
Let $A$, $B$, and $C$ represent real numbers where $A$ and $B$ are not both zero. A **linear equation** in the variables $x$ and $y$ is an equation that can be written as $Ax + By = C$.

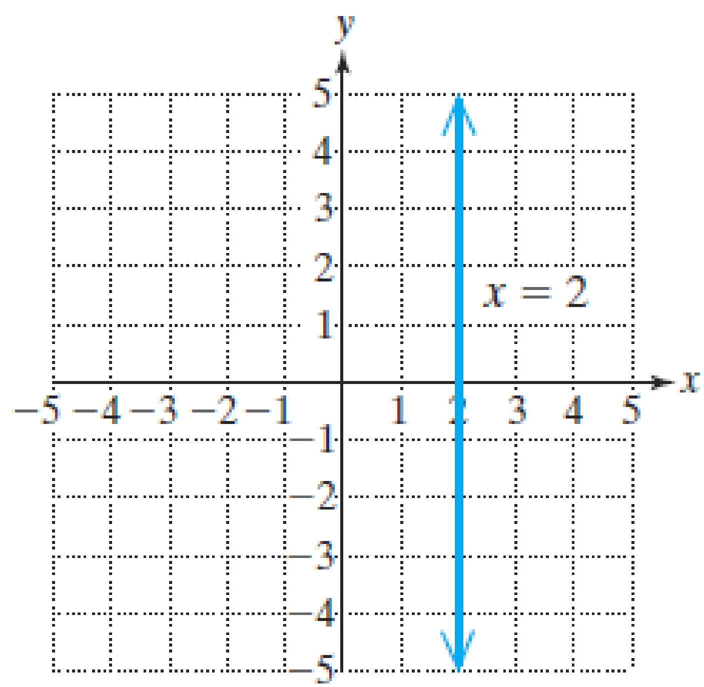
**Example 1:**



The graph of a linear equation is a line.

- A horizontal line has an equation of the form $y = k$ where $k$ is a constant real number.
- A vertical line has an equation of the form $x = k$ where $k$ is a constant real number.



The slope of a line passing through the distinct points $(x_1, y_1)$ and $(x_2, y_2)$ is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

- The slope of a horizontal line is 0.
- The slope of a vertical line is undefined.

**Example 2:**

Given $(-4, 5)$ and $(6, 3)$,

$(x_1, y_1)$ and $(x_2, y_2)$ Label the points.

$$m = \frac{3 - 5}{6 - (-4)} = \frac{-2}{10} = -\frac{1}{5}$$

## Slope-intercept form of a line:

Given a line with slope $m$ and $y$-intercept $(0, b)$, the **slope-intercept form** of the line is given by

$$y = mx + b$$

- A function defined by $f(x) = mx + b$ $(m \neq 0)$ is a **linear function** (graph is a slanted line).
- A function defined by $f(x) = b$ is a **constant function** (graph is a horizontal line).

## Example 3:

Given $2x + 9y = 18$, write the equation in slope-intercept form.

$$2x + 9y = 18$$
$$9y = -2x + 18$$
$$y = -\frac{2}{9}x + 2 \quad \text{or} \quad f(x) = -\frac{2}{9}x + 2$$

The slope is $-\frac{2}{9}$. The $y$-intercept is $(0, 2)$.

## Average rate of change:

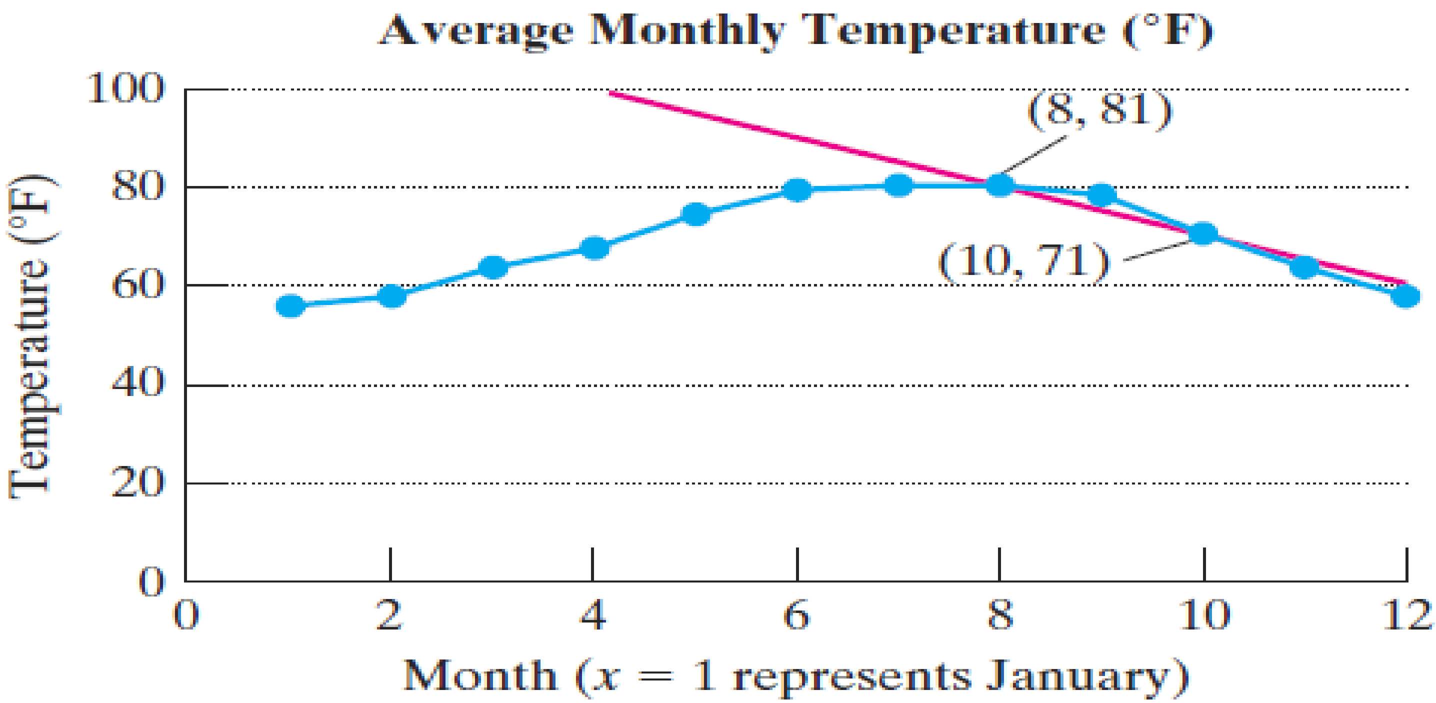
If $f$ is defined on the interval $[x_1, x_2]$, then the **average rate of change** of $f$ on the interval $[x_1, x_2]$ is the slope of the secant line containing $(x_1, f(x_1))$ and $(x_2, f(x_2))$ and is given by

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

**Example 4:**

The average rate of change from $x_1 = 8$ to $x_2 = 10$ is

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{71 - 81}{10 - 8} = \frac{-10}{2} = -5$$




The average monthly temperature between August and October decreased by 5°F per month.

## Solving equations and inequalities graphically:

The $x$-coordinates of the points of intersection between the graphs of $y = f(x)$ and $y = g(x)$ are the solutions to the equation $f(x) = g(x)$.

## Example 5:

Solve the equation and inequalities graphically.

**a.** $2x + 2 = -x + 5$

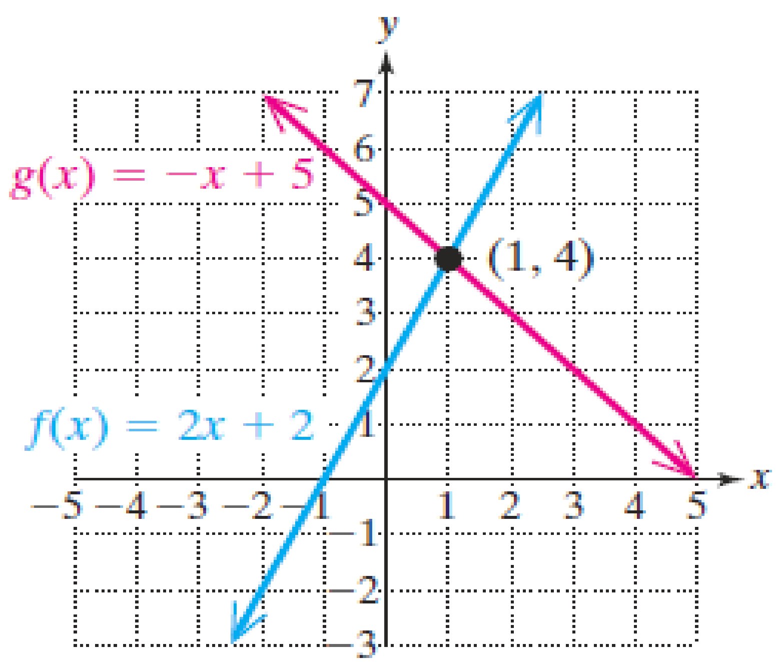
**b.** $2x + 2 < -x + 5$

**c.** $2x + 2 \geq -x + 5$

**a.** $\{1\}$

**b.** $(-\infty, 1)$

**c.** $[1, \infty)$



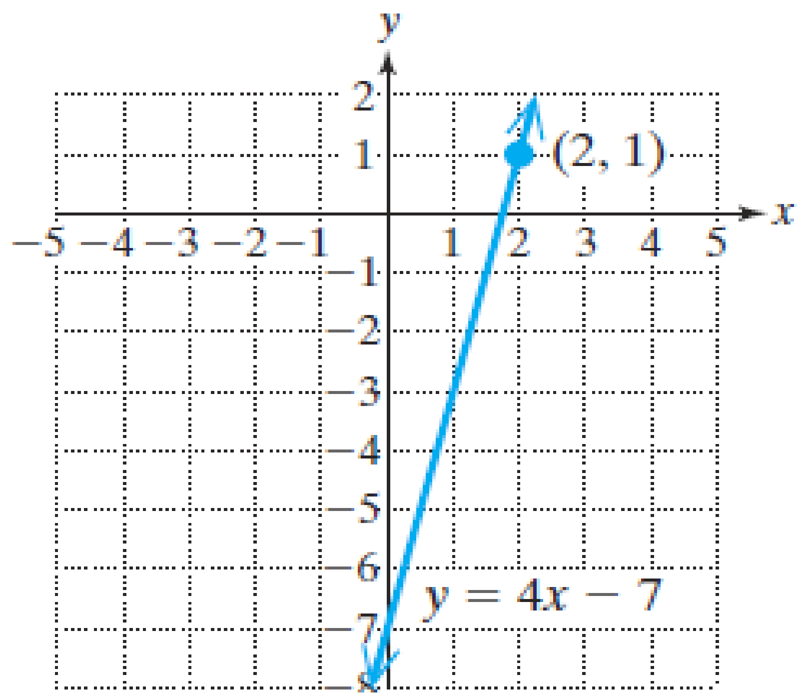
# SECTION 2.5 Applications of Linear Equations and Modeling

The **point-slope formula** for a line is given by $y - y_1 = m(x - x_1)$ where $m$ is the slope of the line and $(x_1, y_2)$ is a point on the line.

**Example 1:**

Use the point-slope formula to find an equation of the line passing through the point (2, 1) and having a slope of 4.

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 4(x - 2)$$
$$y - 1 = 4x - 8$$
$$y = 4x - 7$$



**Slopes of parallel and perpendicular lines:**

- If $m_1$ and $m_2$ represent the slopes of two nonvertical parallel lines, then $m_1 = m_2$.
- If $m_1$ and $m_2$ represent the slopes of two nonvertical perpendicular lines, then $m_1 = -\frac{1}{m_2}$ or equivalently $m_1 m_2 = -1$.

## Example 2:

The slope of a line is $-\frac{1}{5}$.

- The slope of a line parallel to this line is $-\frac{1}{5}$.
- The slope of a line perpendicular to this line is 5.

## EXAMPLE 6 Using a Linear Function in an Application

A family plan for a cell phone has a monthly base price of \$99 plus \$12.99 for each additional family member added beyond the primary account holder.

**a.** Write a linear function to model the monthly cost $C(x)$ (in \$) of a family plan for $x$ additional family members added.

**b.** Evaluate $C(4)$ and interpret the meaning in the context of this problem.

### Solution:

**a.** $C(x) = mx + b$ — The base price \$99 is the fixed cost with zero additional family members added. So the constant $b$ is 99.

$C(x) = 12.99x + 99$ — The rate of increase, \$12.99 per additional family member, is the slope.

**b.** $C(4) = 12.99(4) + 99$ — Substitute 4 for $x$.

$= 150.96$

The total monthly cost of the plan with 4 additional family members beyond the primary account holder is \$150.96.

# SECTION 2.8 Algebra of Functions and Function Composition

## Operations on functions:

Given functions $f$ and $g$, the functions $f + g, f - g, f \cdot g$, and $\frac{f}{g}$ are defined by

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ provided that } g(x) \neq 0$$

The **composition of $f$ and $g$,** denoted $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.

The domain of $f \circ g$ is the set of real numbers $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$.

**Example 1:**

Given $f(x) = x - 3$ and $g(x) = x^2 - 7x$

$$(f + g)(x) = f(x) + g(x) = (x - 3) + (x^2 - 7x)$$
$$= x^2 - 6x - 3$$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{x^2 - 7x}{x - 3} \text{ provided } x \neq 3.$$

**Example 2:**

Given $f(x) = \dfrac{1}{x + 12}$ and $g(x) = x^2 - 7x$

$$(f \circ g)(x) = f(g(x)) = \frac{1}{(g(x)) + 12}$$
$$= \frac{1}{(x^2 - 7x) + 12} = \frac{1}{(x - 3)(x - 4)}$$

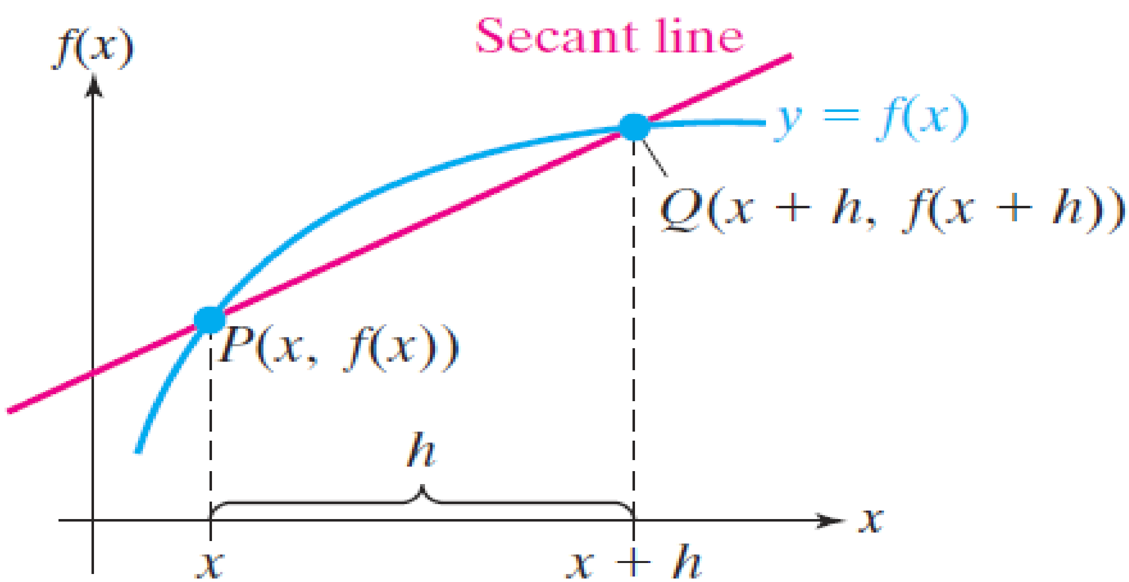
The domain of $f \circ g$ is $\{x \mid x \neq 3 \text{ and } x \neq 4\}$.

In interval notation: $(-\infty, 3) \cup (3, 4) \cup (4, \infty)$.

## Evaluate a difference quotient:

The **difference quotient** represents the average rate of change of a function $f$ between two points $(x, f(x))$ and $(x + h, f(x + h))$.

$$\frac{f(x + h) - f(x)}{h} \quad \text{Difference quotient}$$



**Example 3:**

Given $f(x) = 4x^2 - 5x$ evaluate the difference quotient.

$$\frac{f(x+h) - f(x)}{h}$$

$$= \frac{\overbrace{[4(x+h)^2 - 5(x+h)]}^{f(x+h)} - \overbrace{(4x^2 - 5x)}^{f(x)}}{h}$$

$$= \frac{[4(x^2 + 2xh + h^2) - 5x - 5h] - (4x^2 - 5x)}{h}$$

$$= \frac{4x^2 + 8xh + 4h^2 - 5x - 5h - 4x^2 + 5x}{h}$$

$$= \frac{8xh + 4h^2 - 5h}{h}$$

$$= 8x + 4h - 5$$