الجامعة العربية المفتوحة
Arab Open University

Arab Open University
Faculty of Computer Studies
MT101 - General Mathematics

Chapter 1

Linear Equations and Rational Equations

SECTION 1.1 Linear Equations and Rational Equations

A **linear equation in one variable** is an equation that can be written in the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$.

Example 1:

The equation $2x + 3 = 0$ is a linear equation.

Solving a linear equation in one variable:

1. Simplify both sides of the equation.
2. Collect variable terms on one side of the equation and constants on the other side.
3. Apply the multiplication or division property of equality to obtain a coefficient of 1 on the variable.
4. Check the potential solution.
5. Write the solution set.

Example 2:

Solve.

$$\begin{aligned} -2(x-3)+5x &= 12 \\ -2x+6+5x &= 12 \\ 3x+6 &= 12 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

The solution checks.

The solution set is $\{2\}$.

A **conditional equation** is an equation that is true for some values of the variable but false for others.

A **contradiction** is false for all values of the variable.

An **identity** is true for all values of the variable for which the expressions in the equation are well defined.

Solve a rational equation by multiplying both sides of the equation by the LCD of all fractions in the equation.

Example 3:

$2x = 16$ is a conditional equation because it is true on the condition that $x = 8$. The solution set is $\{8\}$.

$2x + 1 = 2x + 3$ is a contradiction because this statement is false for all values of x. The solution set is $\{\ \}$.

$3x + 6 = 3(x + 2)$ is an identity. It is true for all real numbers, x. The solution set is $\mathbb{R}$.

Example 4:

Solve. $\dfrac{4}{y} = \dfrac{18}{3y} + 1$

$$3y \cdot \left(\frac{4}{y}\right) = 3y \cdot \left(\frac{18}{3y} + 1\right)$$

$$12 = 18 + 3y$$

$$-6 = 3y$$

$$y = -2$$

The solution checks.

The solution set is $\{-2\}$.

SECTION 1.2 Applications and Modeling with Linear Equations

Equations in algebra can be used to organize information from a physical situation. The following are suggested guidelines to solve an application.

1. Read the problem carefully. Assign variables to the unknown quantities.
2. Make an appropriate figure or table, if applicable, and label the given information and variables in the figure or table.
3. Write an equation that represents the verbal model.
4. Solve the equation from step 3.
5. Interpret the solution and check that the answer is reasonable.

Example 1:

How much 20% acid solution should be mixed with 400 mL of a 5% acid solution to bring the concentration rate up to 10%?

Let x represent the amount of 20% solution.

Then $400 + x$ is the amount of 10% solution.

	20% solution	**5% solution**	**10% solution**
Mixture	x	400	$400 + x$
Pure acid	$0.20x$	$0.05(400)$	$0.10(400 + x)$

$$\begin{pmatrix}\text{Acid from} \\ \text{20\% solution}\end{pmatrix} + \begin{pmatrix}\text{Acid from} \\ \text{5\% solution}\end{pmatrix} = \begin{pmatrix}\text{Acid from} \\ \text{10\% solution}\end{pmatrix}$$

$$\begin{aligned} 0.20x + 0.05(400) &= 0.10(400 + x) \\ 0.20x + 20 &= 40 + 0.10x \\ 0.10x &= 20 \\ x &= 200 \end{aligned}$$

200 mL of a 20% acid solution is needed.

Check:

$$\begin{aligned} 0.20(200\text{ mL}) &= 40\text{ mL pure acid} \\ 0.05(400\text{ mL}) &= 20\text{ mL pure acid} \\ \text{Total: } 0.10(600\text{ mL}) &= 60\text{ mL pure acid } \checkmark \end{aligned}$$

SECTION 1.3 Complex Numbers

$i = \sqrt{-1}$ and $i^2 = -1$.

For a real number $b > 0$, $\sqrt{-b} = i\sqrt{b}$.

If a and b are real numbers, then a number written in the form $a + bi$ is called a **complex number.** The value a is called the **real part,** and b is called the **imaginary part.**

To add or subtract complex numbers, combine the real parts, and combine the imaginary parts.

Example 1:

$$\sqrt{-9} \cdot \sqrt{-16} = (3i) \cdot (4i) = 12i^2$$
$$= 12(-1) = -12$$

Example 2:

The number $3 + 8i$ is a complex number with real part 3 and imaginary part 8.

Example 3:

$$(4 + 7i) - (3 + 6i) + (2 + 10i)$$
$$= (4 - 3 + 2) + (7 - 6 + 10)i$$
$$= 3 + 11i$$

Multiply complex numbers by using the distributive property.

Example 4:

$$\begin{aligned}(1-4i)(2+3i) &= 2+3i-8i-12i^2 \\ &= 2-5i-12(-1) \\ &= 2-5i+12 \\ &= 14-5i\end{aligned}$$

Divide complex numbers by multiplying the numerator and denominator by the conjugate of the denominator.

The product of complex conjugates:

$$\begin{aligned}(a+bi)(a-bi) &= (a)^2-(bi)^2 \\ &= a^2-b^2i^2 \\ &= a^2-b^2(-1) \\ &= a^2+b^2\end{aligned}$$

Example 5:

$$\frac{4}{3+5i} = \frac{4 \cdot (3-5i)}{(3+5i)\cdot(3-5i)} = \frac{12-20i}{(3)^2+(5)^2}$$

$$= \frac{12-20i}{9+25} = \frac{12}{34} - \frac{20}{34}i = \frac{6}{17} - \frac{10}{17}i$$

SECTION 1.4 Quadratic Equations

Let a, b, and c represent real numbers. A **quadratic equation** in the variable x is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$.

Zero product property:
If $mn = 0$, then $m = 0$ or $n = 0$.

Square root property:
If $x^2 = k$, then $x = \pm\sqrt{k}$.

Example 1:

Solve. $x^2 - x - 12 = 0$

$(x - 4)(x + 3) = 0$ Factor.

$x - 4 = 0$ or $x + 3 = 0$ Apply the zero product property.

$x = 4$ or $x = -3$

The solution set is $\{4, -3\}$.

Example 2:

Solve. $(x - 3)^2 = 5$

$$x - 3 = \pm\sqrt{5}$$

$$x = 3 \pm \sqrt{5}$$

Solution set: $\{3 \pm \sqrt{5}\}$

Completing the square:

For an equation $ax^2 + bx + c = 0$ $(a \neq 0)$, follow these steps to complete the square and solve the equation.

1. Divide both sides by a to make the leading coefficient 1.
2. Isolate the variable terms on one side.
3. Complete the square by adding the square of one-half the linear term coefficient to both sides. Then factor the resulting perfect square trinomial.
4. Apply the square root property and solve for x.

Example 3:

Solve. $$2x^2 - 8x - 16 = 0$$

$$\frac{2x^2}{2} - \frac{8x}{2} - \frac{16}{2} = \frac{0}{2}$$

$$x^2 - 4x - 8 = 0$$

$$x^2 - 4x + \_\_\_ = 8 + \_\_\_$$

$$x^2 - 4x + 4 = 8 + 4 \quad \textit{Note:} \quad \left[\tfrac{1}{2}(-4)\right]^2 = 4$$

$$(x - 2)^2 = 12$$

$$x - 2 = \pm\sqrt{12}$$

$$x = 2 \pm 2\sqrt{3}$$

Solution set: $\{2 \pm 2\sqrt{3}\}$

Quadratic formula:

The solutions to $ax^2 + bx + c = 0$ ($a \neq 0$) are given by the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant:

The discriminant to the equation $ax^2 + bx + c = 0$ ($a \neq 0$) is given by $b^2 - 4ac$.

The discriminant indicates the number of and type of solutions to the equation.

- If $b^2 - 4ac < 0$, the equation has 2 imaginary solutions.
- If $b^2 - 4ac = 0$, the equation has 1 rational solution.
- If $b^2 - 4ac > 0$, the equation has 2 real solutions.

Example 4:

Solve. $2x^2 - 3x + 7 = 0 \quad a = 2, b = -3, c = 7$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(7)}}{2(2)}$$

$$x = \frac{3 \pm \sqrt{-47}}{4} = \frac{3}{4} \pm \frac{\sqrt{47}}{4}i$$

Solution set: $\left\{\frac{3}{4} \pm \frac{\sqrt{47}}{4}i\right\}$

Example 5:

Determine the number and type of solutions for the equation.

$4x^2 - 5x + 11 = 0$

$a = 4, b = -5, c = 11$

$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4(4)(11) \\ &= -151 \end{aligned}$$

Because $-151 < 0$, the equation has two imaginary solutions.

SECTION 1.5 Applications of Quadratic Equations

Quadratic equations are used to model applications with the Pythagorean theorem, volume, area, and objects moving vertically under the influence of gravity.

The vertical position s of an object moving vertically under the influence of gravity is approximated by $s = -\frac{1}{2}gt^2 + v_0t + s_0$, where

- g is the acceleration due to gravity (at sea level on Earth: $g = 32 \text{ ft/sec}^2$ or 9.8 m/sec^2).
- t is the time after the start of the experiment.
- v_0 is the initial velocity.
- s_0 is the initial position (height).
- s is the position of the object at time t.

Example 1:

A diver jumps approximately straight upward from a diving board 3 m above the water with an initial velocity of 5 m/sec.

a. Write a model to express the height s (in meters) of the diver.

$$s = -\frac{1}{2}(9.8)t^2 + 5t + 3$$

$$s = -4.9t^2 + 5t + 3$$

b. Find the time required for the diver to hit the water.

$$0 = -4.9t^2 + 5t + 3$$

$$x = \frac{-(5) \pm \sqrt{(5)^2 - 4(-4.9)(3)}}{2(-4.9)}$$

$$x = \frac{-5 \pm \sqrt{83.8}}{-9.8} \quad \begin{matrix} \approx -0.42 \text{ sec (reject)} \\ \approx 1.44 \text{ sec} \end{matrix}$$

Rejecting the negative solution, the diver will hit the water approximately 1.44 sec after leaving the board.

SECTION 1.6 More Equations and Applications

Solving polynomial equations:

A polynomial equation with one side equal to zero and the other factored as a product of linear or quadratic factors can be solved by applying the zero product property.

Example 1:

Solve. $8x^3 + 125 = 0$

$$(2x + 5)(4x^2 - 10x + 25) = 0$$

$$2x + 5 = 0 \quad \text{or} \quad 4x^2 - 10x + 25 = 0$$

$$x = -\frac{5}{2} \quad \text{or} \quad x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(4)(25)}}{2(4)}$$

$$x = \frac{10 \pm \sqrt{-300}}{8} = \frac{10 \pm 10i\sqrt{3}}{8}$$

$$x = \frac{5}{4} \pm \frac{5\sqrt{3}}{4}i$$

Solution set: $\left\{-\frac{5}{2}, \frac{5}{4} \pm \frac{5\sqrt{3}}{4}i\right\}$

Solving radical equations:

1. Isolate the radical. If an equation has more than one radical, choose one of the radicals to isolate.
2. Raise each side of the equation to a power equal to the index of the radical.
3. Solve the resulting equation. If the equation still has a radical, repeat steps 1 and 2.
4. Check the potential solutions in the original equation.

Example 2:

Solve. $\sqrt[3]{x-1}+2=6$

$\sqrt[3]{x-1}=4$	Subtract 2 to isolate the radical.
$(\sqrt[3]{x-1})^3=(4)^3$	Cube both sides.
$x-1=64$	Solve for x.
$x=65$	

Solution set: $\{65\}$

Solving an equation of the form $u^{m/n} = k$:

If m is odd:

$$u^{m/n} = k$$
$$(u^{m/n})^{n/m} = (k)^{n/m}$$
$$u = k^{n/m}$$

If m is even:

$$u^{m/n} = k$$
$$(u^{m/n})^{n/m} = \pm(k)^{n/m}$$
$$u = \pm k^{n/m}$$

Example 3:

Solve. $\quad c^{2/5} = 4$

$$(c^{2/5})^{5/2} = \pm(4)^{5/2}$$
$$c = \pm(\sqrt{4})^5$$
$$c = \pm 32$$

Solution set: $\{\pm 32\}$

- **Solving equations in quadratic form:**

Substitution can be used to solve equations that are in quadratic form.

Example 4:

Solve. $x^{2/3} - 3x^{1/3} - 4 = 0$ Let $u = x^{1/3}$.

$$u^2 - 3u - 4 = 0$$

$$(u - 4)(u + 1) = 0$$

$$u = 4 \quad \text{or} \quad u = -1$$

$$x^{1/3} = 4 \quad \text{or} \quad x^{1/3} = -1$$

$$(x^{1/3})^3 = (4)^3 \quad \text{or} \quad (x^{1/3})^3 = (-1)^3$$

$$x = 64 \quad \text{or} \quad x = -1$$

Solution set: $\{64, -1\}$

SECTION 1.7 Linear Inequalities and Compound Inequalities

An inequality that can be written in one of the following forms is a **linear inequality in one variable.**

$ax + b < 0$, $ax + b \leq 0$,
$ax + b > 0$, or $ax + b \geq 0$

***Properties of inequalities:**

Let a, b, and c represent real numbers.

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$, then $a - c < b - c$.
3. If c is *positive* and $a < b$, then

$$a \cdot c < b \cdot c \text{ and } \frac{a}{c} < \frac{b}{c}.$$

4. If c is *negative* and $a < b$, then

$$a \cdot c > b \cdot c \text{ and } \frac{a}{c} > \frac{b}{c}.$$

*These properties of inequality are also true for statements expressed with the symbols $\leq$, $>$, and $\geq$.

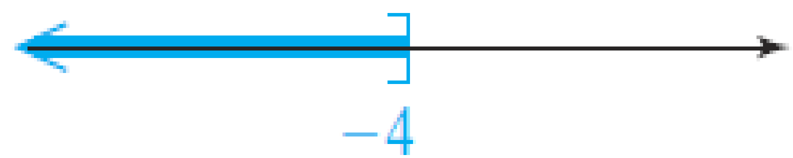
Example 1:

Solve. $-4x - 7 \geq 9$

$$-4x \geq 16$$

$$\frac{-4x}{-4} \leq \frac{16}{-4}$$

$$x \leq -4$$



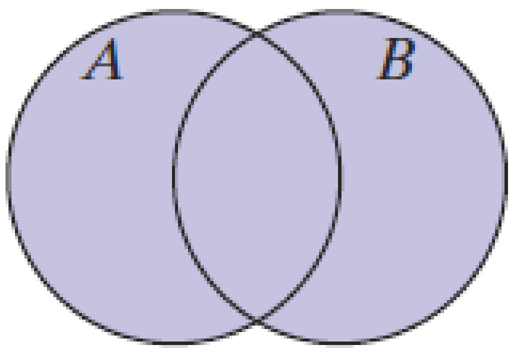
Solution set: $\{x \mid x \leq -4\}$

Interval notation: $(-\infty, -4]$

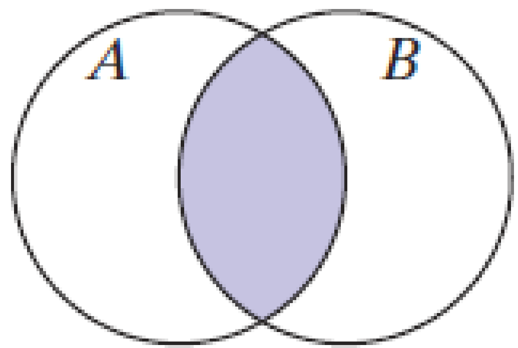
$A \cup B$ is the **union** of A and B. This is the set of elements that belong to set A or set B or to both sets A and B.

$A \cap B$ is the **intersection** of A and B. This is the set of elements common to both A and B.

Example 2:



$A \cup B$



$A \cap B$

Solving compound inequalities:

- If two inequalities are joined by the word "and," take the *intersection* of the individual solution sets.

Example 3:

Solve. $\quad 2x < 6 \quad \text{and} \quad 4 \le x + 5$

$\quad\quad\quad x < 3 \quad \text{and} \quad -1 \le x$

The solution set is $[-1, 3)$.

- The inequality $a < x < b$ is equivalent to $a < x$ and $x < b$.

Example 4:

Solve. $\quad -7 < x + 4 \leq 8$

$$-7 - 4 < x + 4 - 4 \leq 8 - 4$$

$$-11 < x \leq 4$$

The solution set is $(-11, 4]$.

- If two inequalities are joined by the word "or," take the *union* of the individual solution sets.

Example 5:

Solve. $-3y > 12$ or $y + 7 \geq 9$

$y < -4$ or $y \geq 2$

The solution set is $(-\infty, -4) \cup [2, \infty)$.

Properties involving absolute value equations:

Let k represent a real number.

1. If $k > 0$, $|u| = k$ is equivalent to $u = k$ or $u = -k$.

2. If $k = 0$, $|u| = k$ is equivalent to $u = 0$.

3. If $k < 0$, $|u| = k$ has no solution.

4. $|u| = |w|$ is equivalent to $u = w$ or $u = -w$.

Example 1:

Solve. $|x - 3| + 5 = 10$

$|x - 3| = 5$ Isolate the absolute value.

$x - 3 = 5$ or $x - 3 = -5$

$x = 8$ or $x = -2$

The solution set is $\{8, -2\}$.

Example 2:

The equation $|2x+3| = -9$ has no solution.

Example 3:

Solve. $|2x-10| = |x+2|$

$$2x - 10 = x + 2 \quad \text{or} \quad 2x - 10 = -(x+2)$$

$$x = 12 \quad \text{or} \quad 2x - 10 = -x - 2$$

$$x = \frac{8}{3}$$

The solution set is $\left\{12, \frac{8}{3}\right\}$.

***Properties involving absolute value inequalities:**

5. $|u| < k$ is equivalent to $-k < u < k$.

6. $|u| > k$ is equivalent to $u < -k \quad \text{or} \quad u > k$.

*The statements also hold true for the inequality symbols $\leq$ and $\geq$, respectively.

Example 4:

Solve. $|x+7|-3 \le 4$

$$|x+7| \le 7 \qquad \text{Isolate the absolute value.}$$

$$-7 \le x+7 \le 7 \qquad \text{Equivalent form}$$

$$-14 \le x \le 0 \qquad \text{Interval notation: } [-14, 0]$$

Example 5:

Solve. $|x+1| > 5$

$$x+1 < -5 \quad \text{or} \quad x+1 > 5$$

$$x < -6 \quad \text{or} \quad x > 4$$

Interval notation: $(-\infty, -6) \cup (4, \infty)$

Example 6:

$|5x+6| < -4$ has no solution.

$|3y+9| > -3$ is true for all real numbers, $\mathbb{R}$.